Control of Vortex-System Stability

Z. Rusak* and A. Seginer†
Technion—Israel Institute of Technology, Haifa 32000, Israel

A new analytical concept of the control of vortex-systems stability is presented. This approach is based on an analysis of the authors' general theory for the evaluation of the three-dimensional long-wave linear stability of systems of straight vortex filaments immersed in external incompressible potential flowfields in the vicinity of solid surfaces. The introduction of time-dependent geometrical perturbations of the surfaces reshapes the equations of the development of displacement disturbances along the vortices into the form of classic linear-control equations, with the surface perturbations as a controller. The classic criteria of the linear control theory are then applied to adapt the terms of "controllability" and "observability" to vortex-systems stability. Criteria for the analysis of the theoretical possibility of controlling the stability of vortex systems are developed. Based on this approach, the new concept of actively controlling vortex stability in an open- or closed-loop feedback mode is developed. Two examples demonstrate the powerful potential of the combined theories of vortex stability and control. Control laws that are based on measuring vortex disturbances are proposed for the suppression or amplification at will of the well-known Crow instability of a vortex near a straight plane.

Nomenclature

A	= influence matrix, Eq. (24)
A_1	= reduced influence matrix
A_c	= controlled matrix, Eq. (36)
a_{11},a_{12},a_{21}	= components of matrix A, Eq. (30)
$\bar{a}_s$	= forcing function, Eq. (23)
B	= influence matrix of $\hat{r}$, Eq. (24)
$b_{11},b_{12},b_{21},b_{22}$	= components of matrix B, Eq. (30)
$\boldsymbol{b}_1,\boldsymbol{b}_2$	= column vectors of matrix B, Eq. (36)
C	= measurement matrix, Eq. (27)
C^*	= core-diameter parameter, $C^*=\delta/\beta$
C_1,C_2	= components of the measurement matrix C, Eq. (30)
$\bar{C}_{vq}^2$	= generalized damping of a vortex system
c	= vortex core diameter
$\hat{d}$	= disturbance vector, Eq. (24)
$\hat{d}_1,\hat{d}_2$	= components of vector $\hat{d}$, Eq. (30)
$\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z$	= unit vectors in the (x,y,z) directions respectively
F_R	= general description of the solid surfaces, $F_R=0$
f_R	= general description of the solid surfaces' basic state
$\mathfrak{F}$	= kernel function, Eq. (23)
H	= control functional or matrix, Eq. (34)
h	= distance of the vortex from the plane
I_1	= special function in Eq. (38)
I_{c1}	= special function in Eq. (37)
i	= imaginary unit, $i=\sqrt{-1}$
k	= wave number, Eq. (21)
ℓ	= characteristic length
N	= number of vortices
P	= controllability matrix, Eq. (28)
Q	= observability matrix, Eq. (29)
$\mathfrak{Q}$	= source distribution function, Eq. (4)
$\mathfrak{Q}_0$	= source distribution basic state
$\mathfrak{Q}_1$	= source distribution fluctuation
$\bar{\mathfrak{Q}}_{ym},\bar{\mathfrak{Q}}_{zm},\bar{\mathfrak{Q}}_{R1},\bar{\mathfrak{Q}}_{\dot{R}1}$	= source distribution functions
R_0	= radius of circular tube
R_1	= small surface perturbations, $R_1(x_R,t)$
$\hat{R}_1$	= Fourier transform of R_1, Eq. (24)
$\boldsymbol{R}_n$	= position vector, $\boldsymbol{R}_n(S_n,t)$, describing the nth vortex line
$\boldsymbol{R}_p$	= position vector to a general point in the flowfield
$\boldsymbol{R}_R$	= position vector, $\boldsymbol{R}_R(x,y,z,t)$, describing a point on the solid surfaces
r	= radial distance, Eqs. (26)
$\hat{r}$	= control vector, Eq. (24)
S	= Laplace variable
S_n	= point along the nth vortex line
t	= time
t_0,t_1	= initial and terminal time
$\boldsymbol{U}$	= external flowfield velocity
U	= x-wise component of external flow $\boldsymbol{U}$, Eq. (17)
U_∞	= velocity field at infinity
$\boldsymbol{u}_n,\boldsymbol{u}_p,\boldsymbol{u}_R$	= velocity vectors
V	= y-wise component of external flow $\boldsymbol{U}$, Eq. (17)
W	= z-wise component of external flow $\boldsymbol{U}$, Eq. (17)
$W(\delta)$	= self-induction function, Eq. (37)
x_n	= x-wise coordinate along the nth vortex
x_R	= x-wise coordinate along the solid surfaces
$\hat{y}_{c1}$	= component of vector $\hat{y}_c$, Eq. (30)
y_n	= y-wise displacement wave of the nth vortex, Eq. (10)
y_{0n}	= y-wise coordinate of vortex basic-state position, Eq. (10)
y_R	= y-wise coordinate of a point on $F_R=0$
$\hat{y}$	= displacement vector, Eq. (24) (Fourier-transformed)
$\hat{y}_0$	= displacement vector at $t=t_0$
$\hat{y}_c$	= measurement vector, Eq. (27)
$\hat{y}_1,\hat{z}_1$	= components of $\hat{y}$, Eq. (30)
z_n	= z-wise displacement wave of the nth vortex, Eq. (10)

Received Sept. 20, 1989; revision received July 3, 1990; accepted for publication Aug. 1, 1990.

*Graduate Student, Faculty of Aerospace Engineering; currently Postgraduate Fellow, Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180.

†Professor, Faculty of Aerospace Engineering. Associate Fellow AIAA.

z_{0n} = z-wise coordinate of vortex basic-state position, Eq. (10)
z_R = z-wise coordinate of a point on $F_R = 0$
β = dimensionless wave number
Γ, Γ_n = circulation
$\lambda_{1,2}$ = eigenvalues
δ = cut-off distance, $\delta = C^*\beta$, also perturbation displacements in Eqs. (15)
$\chi(\beta), \psi(\beta)$ = Crow's interaction functions
ϕ = velocity potential, Eq. (1)
θ_{R0} = Lagrangian parameter of a point on $f_R = 0$ at a specific cross section x_R

Subscripts

m, n = vortex index number
$n0$ = index, Eq. (20a)
p = index of a point in the flowfield
q = index, Eq. (40)
R = surface index
$R0$ = index, Eq. (20b)
S = index, Eq. (23)
x = x-wise component or derivative
y = y-wise component or derivative
z = z-wise component or derivative
0 = basic-state condition

Superscripts

S = influence of sources
V = influence of vortices
$\hat{}$ = Fourier-transformed
$'$ = integration variable, Eq. (23) and (26)

Introduction

MODERN flight vehicles, and especially highly maneuverable modern fighter aircraft and missiles, generate highly complex, three-dimensional flowfields that contain concentrated vortices. The vortical flows strongly affect the aerodynamic characteristics of the vehicles. In the case of military maneuverable vehicles, as well as in several novel designs of general aviation vehicles, the vortices contribute to vehicle performance and extend its operational flight envelope to super maneuverability and increased agility.[1] In more conventional aircraft, such as large transports, the vortical wakes induce drag on the aircraft and can endanger small vehicles that inadvertently happen to cross these wakes.[2]

The utilization of the vortex fields to enhance flight-vehicle performance is strongly limited by vortex instabilities. These are usually accompanied by strong unsteady phenomena that can endanger the stability of the flight vehicle itself.[1] On the other hand, destabilization of the tip vortices of heavy transport aircraft could reduce their drag penalty and alleviate the air-traffic controllers' problem of keeping small craft out of the wakes of larger ones.[2] The ability to predict the flowfield conditions that destabilize the vortices, and even better, the ability to control this stability would, therefore, be beneficial for the future utilization of vortex flows in the aerodynamic design of flight vehicles. Future vortex-stability control (either stabilization or destabilization) could greatly enhance the aerodynamic performance of flight vehicles.

All of the applications of the classic approach to vortex-stability analysis are studies of specific self-preserving vortex flowfield configurations. These include, for example, the two-dimensional cases of point vortices in steady or quasisteady flowfields under two-dimensional perturbations, e.g., Refs. 3–9, or the few studies of basically two-dimensional flows with three-dimensional perturbations, e.g., see Refs. 10–16. The classic approach is to study the linear stability of the perturbed vortex system by determining the character of the eigenvalues of the linearized equations that describe the development of the perturbations. Only recently has the classic approach been applied by the present authors[17,18] to the three-dimensional long-wave linear stability of a general system of straight concentrated vortices in the vicinity of solid surfaces and in an external incompressible potential flow. Using this general theory, the authors of this paper presented a new approach to the vortex-stability problem.[19] It proposes a new physical model that identifies basic physical mechanisms that drive dynamic and static instabilities in vortex systems. This model is based on an analogy between the development of the three-dimensional perturbations to the vortex filaments and the motion of a mechanical system of a unit point mass on a spring and dash-pot combination.

Despite the large number of analyses of two-dimensional stability problems[3-9] and several three-dimensional studies,[10-16] the prediction of the observability and controllability of vortex systems by the classic linear-control approach has never been proposed. Such a prediction could lead to the measurement of the development of the perturbations in vortex systems and to their control. Controlling vortex instabilities could open the gate to a new technology of stabilizing or destabilizing vortices at will.

Presented in this paper is a new analytical approach to the question of vortex-stability control.[20] It is based on the general theory of Rusak and Seginer[18] and on the well-established criteria of controllability and observability, defined by linear-control theory.[24] With these criteria, the theoretical possibility of controlling a vortex-system stability can be analyzed. The following analysis is limited to steady or time-dependent incompressible potential flowfields with infinitely long thin straight concentrated vortices parallel to infinitely long bodies. The stability analysis is limited to small three-dimensional long-wave perturbations.

Two examples are presented to demonstrate the powerful potential of the combined theories of vortex stability and vortex control. The challenging concept of active control of vortex stability is proposed for future research.

Mathematical Model

The model developed by Rusak and Seginer[18] considers a configuration of N potential vortex filaments with different, but known circulations $\Gamma_n (n = 1, \ldots, N)$, in an incompressible potential external flowfield. The velocity-potential field (ϕ) of the flow, $\boldsymbol{u} \equiv \nabla\phi$, is described by the Laplace equation

$$\nabla^2\phi = 0 \tag{1}$$

The solution of Eq. (1) is governed by the streak-line equations that describe each of the N vortex lines $\boldsymbol{R}_n(S_n, t)$

$$\frac{\mathrm{D}\boldsymbol{R}_n}{\mathrm{D}t} = \boldsymbol{u}_n(S_n, t) \qquad (n = 1, \ldots, N) \tag{2}$$

where $\boldsymbol{R}_n(S_n, t)$ is the position vector to point S_n on the nth vortex filament at time t, and $\boldsymbol{u}_n(S_n, t)$ is the velocity induced by the flowfield on this point.

When solid but flexible surfaces, given by their time-dependent shapes $F_R(\boldsymbol{R}_R, t) = 0$, also are present in the flowfield (where $\boldsymbol{R}_R$ is the position vector to a point on the surface), the flow-tangency boundary condition has to be satisfied at each point $\boldsymbol{R}_R$ and time t

$$\frac{\mathrm{D}F_R}{\mathrm{D}t} \equiv \frac{\partial F_R}{\partial t} + \boldsymbol{u}_R \cdot \nabla F_R = 0 \qquad \text{on} \qquad F_R = 0 \tag{3}$$

where $\boldsymbol{u}_R$ is the velocity induced by the flowfield on any point on the surfaces, and ∇F_R is a vector normal to the surface $F_R = 0$, pointing into the flowfield.

To satisfy the tangency condition (3), the solid surfaces $F_R = 0$ are represented by a potential source distribution $\mathcal{Q}$ over them

$$\mathcal{Q} = \mathcal{Q}(\boldsymbol{R}_R, t) \tag{4}$$

In unbounded flowfields, the velocities induced by either the vortices or the solid surfaces vanish at large distances r from them, like $1/r$ and

$$\nabla\phi \to \boldsymbol{U}_\infty \quad \text{as} \quad r\to\infty \tag{5}$$

where $\boldsymbol{U}_\infty$ is a given flow velocity vector at infinity. Initial conditions that describe the vortex lines at time $t=t_0$ are also given for every n by $\boldsymbol{R}_n(S_n,t_0)$ $(n=1,\ldots,N)$. The solution of Eqs. (1) through (5) gives the equilibrium dynamics of the vortices in the flowfield, referred to here as the "basic state."

The total velocity $\boldsymbol{u}_p$ at a point $\boldsymbol{R}_p$ in the flowfield is given by the sum

$$\boldsymbol{u}_p = \boldsymbol{U} + \boldsymbol{u}_p^V + \boldsymbol{u}_p^S \tag{6}$$

where $\boldsymbol{U}$ is a given external incompressible potential flowfield satisfying

$$\nabla\cdot\boldsymbol{U}=0, \qquad \nabla\times\boldsymbol{U}=0 \tag{7}$$

In Eq. (6), $\boldsymbol{u}_p^V$ is the velocity induced by all the vortex lines on a point $\boldsymbol{R}_p$ and is given by the Biot–Savart law,[22]

$$\boldsymbol{u}_p^V = \sum_{m=1}^{N}\frac{\Gamma_m'}{4\pi}\int_{R_m'}\frac{\boldsymbol{R}_{mp}'\times \mathrm{d}\boldsymbol{R}_m'}{|\boldsymbol{R}_{mp}'|^3} \tag{8}$$

in terms of the relative positions $\boldsymbol{R}_{mp}'\equiv\boldsymbol{R}_m'-\boldsymbol{R}_p$, length elements $\mathrm{d}\boldsymbol{R}_m'$, and circulation intensities Γ_m'. When point $\boldsymbol{R}_p$ lies on a vortex filament, then $p=n$. The primes are used here to designate points on a vortex filament and terms related to them. The logarithmic singularity when $p=n$ in the calculation of the self-induced velocity of a curved vortex line is treated by the "cut-off distance" model of Widnall[23] or of Moore and Saffman.[24]

The variable $\boldsymbol{u}_p^S$ is the velocity induced by the source distribution $\mathcal{Q}$ on a point $\boldsymbol{R}_p$ and is given by[22]

$$\boldsymbol{u}_p^S = -\iint_{F_R'}\frac{\mathcal{Q}'\boldsymbol{R}_{Rp}'\,\mathrm{d}F_R'}{4\pi|\boldsymbol{R}_{Rp}'|^3} \tag{9}$$

in terms of the relative position $\boldsymbol{R}_{Rp}'=\boldsymbol{R}_R'-\boldsymbol{R}_p$, the area of the surface element of $\mathrm{d}F_R'$, and the source strength $\mathcal{Q}'$ at a point $\boldsymbol{R}_R'$. When point $\boldsymbol{R}_p$ lies on the surface, then $p=R$. The primes are used here to designate points lying on the surfaces $F_R=0$ and quantities related to them.

An orthogonal and inertial Cartesian coordinate system (x,y,z) with unit vectors $(\boldsymbol{e}_x,\boldsymbol{e}_y,\boldsymbol{e}_z)$, respectively, is used to describe the flowfield. The N vortices are assumed to be infinite filaments, each along its x_n axis (which is parallel to the x axis), and described by

$$\boldsymbol{R}_n=\boldsymbol{e}_x x_n+\boldsymbol{e}_y(y_{0n}+y_n)+\boldsymbol{e}_z(z_{0n}+z_n) \quad (n=1,\ldots,N) \tag{10}$$

where x_n is the Lagrangian variable, $x_n\equiv S_n$ that goes from $(-\infty)$ to $(+\infty)$, (y_{0n},z_{0n}) are the basic-state (unperturbed) position coordinates of the nth vortex that depend on time t only, $y_{0n}=y_{0n}(t)$, $z_{0n}=z_{0n}(t)$, and (y_n,z_n) are two orthogonal displacement waves in the y and z directions, respectively, that are imposed on each of the vortex filaments and that depend on position (x_n) along the nth vortex and on time t, $y_n=y_n(x_n,t)$, $z_n=z_n(x_n,t)$. The initial conditions at time t_0 are assumed to be given for every n by

$$y_{0n}(t_0),\quad z_{0n}(t_0),\quad y_n(x_n,t_0),\quad z_n(x_n,t_0) \tag{11}$$

The solid surfaces $F_R=0$ are defined as infinite surfaces along an x_R axis (that also is parallel to the x axis), and given at each point $\boldsymbol{R}_R=\boldsymbol{e}_x x_R+\boldsymbol{e}_y y_R+\boldsymbol{e}_z z_R$ by

$$F_R(x_R,y_R,z_R,t)=f_R(y_R,z_R)-R_1(x_R,t)=0 \tag{12}$$

where x_R goes from $(-\infty)$ to $(+\infty)$, $f_R(y_R,z_R)=0$ is the basic-state form of the surfaces $F_R=0$, defined as infinite two-dimensional surfaces, and $R_1(x_R,t)$ is an a priori known unsteady lengthwise perturbation to the shape of the surfaces $f_R=0$. The functions f_R and R_1 are given as dimensionless functions. The characteristic length ℓ of the problem is defined as the minimal distance between the vortices in their basic state or between these vortices and the surface $f_R=0$

$$\ell=\min_t\left\{\min_{\substack{R,n\\ m\neq n}}\sqrt{(y_{0m}-y_{0n})^2+(z_{0m}-z_{0n})^2},\ \min_{R,n}\sqrt{(y_R-y_{0n})^2+(z_R-z_{0n})^2}\right\} \tag{13}$$

All the perturbations are characterized by sufficiently small amplitudes and slopes compared with the characteristic length ℓ of the problem, for every n and (x_n,t)

$$\left(\frac{y_n}{\ell}\right)^2\ll 1,\quad \left(\frac{z_n}{\ell}\right)^2\ll 1,\quad \left(\frac{\partial y_n}{\partial x_n}\right)^2\ll 1,\quad \left(\frac{\partial z_n}{\partial x_n}\right)^2\ll 1 \tag{14a}$$

and for every (x_R,t)

$$R_1^2\ll 1,\qquad \ell^2\left(\frac{\partial R_1}{\partial x_R}\right)^2\ll 1 \tag{14b}$$

Each point on the surfaces $F_R=0$ is assumed to be given at a specific cross section x_R by

$$y_R=y_{R0}+\delta y_R,\qquad z_R=z_{R0}+\delta z_R \tag{15}$$

where (y_{R0},z_{R0}) are the (y,z) coordinates of a point on the basic-state surfaces, $f_R(y_{R0},z_{R0})=0$, that are also represented by a Lagrangian parameter θ_{R0}, $y_{R0}=y_{R0}(\theta_{R0})$, $z_{R0}=z_{R0}(\theta_{R0})$, and $(\delta y_R,\delta z_R)$ are small displacements that depend on the perturbation $R_1(x_R,t)$.

The source distribution $\mathcal{Q}$ also is assumed to be constructed of a basic-state term $\mathcal{Q}_0$ that depends on (y_{R0},z_{R0},t), and a fluctuation $\mathcal{Q}_1$ that depends on (x_R,y_R,z_R,t)

$$\mathcal{Q}=\mathcal{Q}_0(y_{R0},z_{R0},t)+\mathcal{Q}_1(x_R,y_R,z_R,t) \tag{16a}$$

where for every n and $(\boldsymbol{R}_R,t)$,

$$\left(\frac{\mathcal{Q}_1}{\Gamma_n/2\pi\ell}\right)^2\ll 1 \tag{16b}$$

The external flowfield $\boldsymbol{U}$ is described by

$$\boldsymbol{U}=\boldsymbol{e}_x U+\boldsymbol{e}_y V(y,z)+\boldsymbol{e}_z W(y,z) \tag{17}$$

where U is a uniform axial velocity and V and W are steady cross-flow velocities in the y and z directions. This is in principle a quasi-two-dimensional flow, but despite the dynamic similarity, it can be shown that under certain conditions the U component has a major influence on the static stability of the vortex system (for details, see Ref. 18).

Substituting Eqs. (6) and (10) in the streak-line equations, Eqs. (2), results for every n and (x_n,t) in

$$\frac{\mathrm{d}y_{0n}}{\mathrm{d}t}+\frac{\partial y_n}{\partial t}+\left[U+\left(u_n^V\right)_x+\left(u_n^S\right)_x\right]\frac{\partial y_n}{\partial x_n} = V(y_{0n}+y_n,z_{0n}+z_n)+\left(u_n^V\right)_y+\left(u_n^S\right)_y$$

$$\frac{\mathrm{d}z_{0n}}{\mathrm{d}t}+\frac{\partial z_n}{\partial t}+\left[U+\left(u_n^V\right)_x+\left(u_n^S\right)_x\right]\frac{\partial z_n}{\partial x_n} = W(y_{0n}+y_n,z_{0n}+z_n)+\left(u_n^V\right)_z+\left(u_n^S\right)_z \tag{18}$$

Substituting Eqs. (6) and (12) in the flow-tangency boundary condition, Eq. (3), results for every $(\boldsymbol{R}_R, t)$ in

$$
\begin{aligned}
-\frac{\partial R_1}{\partial t} &+ \left[U + (u_R^V)_x + (u_R^S)_x\right] F_{Rx} \\
&+ \left[V(y_R, z_R) + (u_R^V)_y + (u_R^S)_y\right] F_{Ry} \\
&+ \left[W(y_R, z_R) + (u_R^V)_z + (u_R^S)_z\right] F_{Rz} = 0
\end{aligned} \tag{19}
$$

where F_{Rx}, F_{Ry}, F_{Rz} are the (x, y, z) components of vector ∇F_R.

In the basic (unperturbed) state, the N vortices are assumed to be straight parallel filaments (for every n: $y_n = z_n = 0$), and the solid surfaces are defined as infinite two-dimensional steady surfaces $f_R = 0$ (for every $\boldsymbol{R}_R$: $\mathcal{Q}_1 = R_1 = 0$). Equations (18) and (19) result then in the basic-state or zero-order equations for every (n, t),

$$
\begin{aligned}
\frac{\mathrm{d}y_{0n}}{\mathrm{d}t} &= V(y_{0n}, z_{0n}) + v_{n0}^V + v_{n0}^S \\
\frac{\mathrm{d}z_{0n}}{\mathrm{d}t} &= W(y_{0n}, z_{0n}) + w_{n0}^V + w_{n0}^S
\end{aligned} \tag{20a}
$$

and for every (θ_{R0}, t),

$$
\begin{aligned}
&\left[V(\theta_{R0}) + v_{R0}^V + v_{R0}^S\right] f_{Ry0} + \left[W(\theta_{R0}) + w_{R0}^V + w_{R0}^S\right] f_{Rz0} \\
&+ \tfrac{1}{2}\, \mathcal{Q}_0(\theta_{R0}, t) = 0
\end{aligned} \tag{20b}
$$

where v_{n0}^V, v_{n0}^S, v_{R0}^V, v_{R0}^S and w_{n0}^V, w_{n0}^S, w_{R0}^V, w_{R0}^S are the y and z basic-state components of the velocities u_n^V, u_n^S, u_R^V, u_R^S, respectively. Also in Eqs. (20), (f_{Ry0}, f_{Rz0}) are the (y, z) components of the unit vector $\nabla f_R / |\nabla f_R|$. The solution of Eqs. (20) gives the two-dimensional dynamics of the vortices in the flowfield, $y_{0n}(t)$ and $z_{0n}(t)$, for every n, together with the basic-state source distribution $\mathcal{Q}_0(\theta_{R0}, t)$ over the surfaces $f_R = 0$.

A linearization of the flowfield equations, Eqs. (8), (9), (12), (16–19), with respect to the basic state, results[18] in a first-order system of linear integro-differential equations for the solution of the vortex perturbations $y_n(x_n, t)$, $z_n(x_n, t)$ and the source fluctuations $\mathcal{Q}_1(x_R, \theta_{R0}, t)$. The perturbations are assumed to be represented by Fourier integrals

$$
\begin{aligned}
y_n(x_n, t) &= \int_{-\infty}^{+\infty} \hat{y}_n(k, t) e^{ikx_n}\, \mathrm{d}k \\
z_n(x_n, t) &= \int_{-\infty}^{+\infty} \hat{z}_n(k, t) e^{ikx_n}\, \mathrm{d}k \\
\mathcal{Q}_1(x_R, \theta_{R0}, t) &= \int_{-\infty}^{+\infty} \hat{\mathcal{Q}}_1(k, \theta_{R0}, t) e^{ikx_R}\, \mathrm{d}k \\
R_1(x_R, t) &= \int_{-\infty}^{+\infty} \hat{R}_1(k, t) e^{ikx_R}\, \mathrm{d}k
\end{aligned} \tag{21}
$$

where k is the wave number of a sinusoidal perturbation in the x direction, $i = \sqrt{-1}$, and $\hat{y}_n, \hat{z}_n, \hat{\mathcal{Q}}_1, \hat{R}_1$ are the Fourier transforms of $y_n, z_n, \mathcal{Q}_1, R_1$, respectively. Also $\hat{\mathcal{Q}}_1$ is assumed to be a linear combination of the Fourier transforms of the vortex and surface perturbations

$$
\hat{\mathcal{Q}}_1 = \sum_{m=1}^{N} \frac{\Gamma_m}{\pi} (\hat{y}_m \bar{\mathcal{Q}}_{ym} - \hat{z}_m \bar{\mathcal{Q}}_{zm}) + \hat{R}_1 \bar{\mathcal{Q}}_{R1} + \frac{\partial \hat{R}_1}{\partial t} \bar{\mathcal{Q}}_{\dot{R}1} \tag{22}
$$

where the functions $\bar{\mathcal{Q}}_{ym}, \bar{\mathcal{Q}}_{zm}, \bar{\mathcal{Q}}_{R1}, \bar{\mathcal{Q}}_{\dot{R}1}$ are unknown source distribution functions. Substituting Eqs. (21) and (22) in the linearized integro-differential equations results after an involved but straightforward analysis,[18] for every dimensionless wave number $\beta \equiv k\ell$ of the imposed perturbations, in a system of $(N+2)$ Fredholm integral equations of the second kind for the solution of the functions $\bar{\mathcal{Q}}_S$ $(S = ym, zm, R1, \dot{R}1)$

$$
\begin{aligned}
&\bar{\mathcal{Q}}_S(\beta, \theta_{R0}, t) + \int_{\theta_{RL}}^{\theta_{RU}} \bar{\mathcal{Q}}_S(\beta, \theta_{R0}', t) \mathfrak{F}(\beta, \theta_{R0}', \theta_{R0}, t)\, \mathrm{d}\theta_{R0}' \\
&= \bar{a}_S(\beta, \theta_{R0}, t)
\end{aligned} \tag{23}
$$

and in a system of first-order, linear differential equations that describe the time history of the Fourier-transformed displacements of the vortices

$$
\frac{\partial \hat{y}}{\partial t} = A(\beta, t) \hat{y}(\beta, t) + B(\beta, t) \hat{r}(\beta, t) + \hat{d}(\beta, t) \tag{24}
$$

In Eqs. (23) the kernel function $\mathfrak{F}$ and forcing functions $\bar{a}_S$ are functions of β and of the basic-state solutions, and $(\theta_{RU}, \theta_{RL})$ are the values of θ_{R0} at the boundaries of the surfaces $f_R = 0$, when they are described by θ_{R0}. In Eq. (24), $\hat{y}$ is a $2N$-dimensional column vector of the Fourier-transformed perturbations, $\hat{y}^T = \hat{y}^T(\beta, t) = \{\ldots, \hat{y}_n(\beta, t), \hat{z}_n(\beta, t), \ldots\}$, A is a $2N \times 2N$-dimensional influence matrix, $\hat{r}$ is a 2×1-dimensional vector of the Fourier transform of $(R_1, \partial R_1/\partial t)$, B is its $2N \times 2$-dimensional influence matrix, and $\hat{d}$ can be any other general disturbance vector or a vector that arises from specific initial conditions. The components of matrices A, B are functions of the wave number β, the basic-state solution, and the source distribution functions $\bar{\mathcal{Q}}_S$. The matrix A is a function also of the N dimensionless cutoff distances δ_n of each of the N vortices, which are given by

$$
\delta_n = C_n^* \beta \tag{25}
$$

where C_n^* is the nth vortex core-diameter parameter that can be evaluated by the Widnall[23] or Moore and Saffman[24] models of the cutoff distance

$$
C_n^* = \frac{c_n/2}{\ell} \exp[1/2 - \ell n\, 2 - A_n + C_n] \tag{26a}
$$

In Eq. (26a)

$$
\begin{aligned}
A_n &= \lim_{2r/c_n \to \infty} \left[\int_0^{2r/c_n} r' V_{0n}^2\, \mathrm{d}r' - \ell n \frac{2r}{c_n} \right] \\
C_n &= \frac{8}{c_n^2} \int_0^{\infty} r' W_{0n}^2\, \mathrm{d}r'
\end{aligned} \tag{26b}
$$

where

$$
V_{0n} = \frac{V_{\theta n}(r')}{\Gamma_n / \pi c_n}, \qquad W_{0n} = \frac{V_{xn}(r')}{\Gamma_n / \pi c_n}
$$

where c_n is the core diameter of the nth vortex, r is the radial distance from the vortex axis, $r' = 2r/c_n$, and (V_{0n}, W_{0n}) are the dimensionless circumferential and axial velocity distributions $(V_{\theta n}, V_{xn})$, respectively, in the nth vortex rotational core. The use of the "cutoff distance" model limits the present theory to slender vortices where $(c_n/\ell)^2 \ll 1$ or $C_n^{*2} \ll 1$ and to long-wave perturbations where the dimensionless wave number of a sinusoidal perturbation is limited by $(\beta C_n^*)^2 \ll 1$. For details of the analysis, as well as for the functions $\mathfrak{F}$, $\bar{a}_S$ and the components of the matrices A, B, see Ref. 18.

In the general case of an unsteady basic state of the vortex motion, the matrices A and B are time-dependent and their components are determined by the basic-state solution. The solution of Eq. (24) in this case is quite difficult and can be obtained in most cases only numerically. On the other hand, when the basic state is steady or quasisteady, the matrices A and B are time-invariant and Eq. (24) can be solved analytically.

It can be shown[17,18] that the matrix A can be written in the general case in the form

$$A \equiv A_1 - i\frac{\beta}{\ell}UI$$

where A_1 is a $2N \times 2N$-dimensional reduced influence matrix with a vanishing trace [trace $(A_1)=0$]. It also can be shown that each of the stability analyses studied in Refs. 3–16 can be described as a particular case of Eq. (24), without a perturbation R_1 of the surfaces and without a disturbance vector $\hat{d}$.

Control-Theory Approach

When an equation for the output $\hat{y}_c$ of the measurement of $\hat{y}$ at every β

$$\hat{y}_c(\beta,t) = C(\beta,t)\hat{y}(\beta,t) \tag{27}$$

is included, Eqs. (24) and (27) have the general form of the basic equations of the classic linear-control theory,[21] where $\hat{y}$ is the state variable vector, $A(\beta,t)$ is the system matrix, $\hat{r}(\beta,t)$ is the input or control vector, $B(\beta,t)$ is the control matrix, $\hat{y}_c$ is the measurement vector, and $C(\beta,t)$ is the measurement matrix. Following Kwakernaak and Sivan,[21] the concepts of "controllability" and "observability" can now be defined:

1) A vortex system is defined as "controllable" at a certain wave number β if the disturbances' state vector $\hat{y}(\beta,t)$ can be transferred from its initial state at any time t_0 to any terminal state at time t_1 within a finite time interval $t_1 - t_0$.

2) A vortex system is defined as "observable" (or "reconstructible") at a certain wave number β if for every time t_1 there exists a time t_0 within the interval $-\infty < t_0 < t_1$ such that for all $\hat{r}(\beta,t)$ and $t_0 \le t \le t_1$, the condition $\hat{y}_c[t,t_0,\hat{y}_0(\beta,t_0),\hat{r}] = \hat{y}_c[t,t_0,\hat{y}_0'(\beta,t_0),\hat{r}]$ implies that $\hat{y}_0 = \hat{y}_0'$ [where $y_0(\beta,t_0)$ is the initial condition at time t_0].

Controllability means that the perturbations along the vortex lines can, for a given β, be steered from one given state to any other state. It can be shown[17] that a vortex system that is steady in its basic state is controllable at a certain wave number β if, and only if, the rank $[P(\beta)]$ of the controllability matrix $P(\beta)$ equals $2N$, where

$$P(\beta) = \left[B, AB, \ldots, A^{2N-1}B\right] \tag{28}$$

In this case, the unstable modes of the disturbances to the vortex filaments can theoretically be stabilized. Following Kwakernaak and Sivan,[21] a controllability criterion can be found also for the general case where the basic state is unsteady.[17]

Observability means that the behavior of all the disturbances' state vector $\hat{y}$ can be determined by the behavior of the output-measurement vector $\hat{y}_c$. Observability also includes the determination of the minimum number of parameters that have to be measured for the reconstruction of the behavior of the perturbations. It can be shown[17] that a vortex system that is steady in its basic state is observable if, and only if, the rank $[Q(\beta)]$ of the observability matrix $Q(\beta)$ equals $2N$, where

$$Q(\beta) = \left[C^T, A^TC^T, \ldots, (A^{2N-1})^TC^T\right] \tag{29}$$

Following Kwakernaak and Sivan,[21] an observability criterion can be derived also for the general case where the basic state is unsteady.[17]

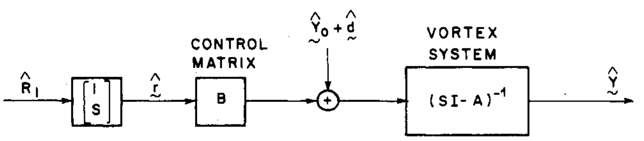

Fig. 1 Open-loop control.

When the problem described by Eq. (24) can be reduced to a two-rank problem, the time-invariant matrices $A(\beta)$ and $B(\beta)$ are 2×2 dimensional and the matrix $C(\beta)$ is 1×2 dimensional. The terms in Eq. (24) can then be written in the following form:

$$A(\beta) = \begin{bmatrix} -i\frac{\beta}{\ell}U + a_{11} & a_{12} \\ a_{21} & -i\frac{\beta}{\ell}U - a_{11} \end{bmatrix}, \quad B(\beta) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} \hat{y}_1 \\ \hat{z}_1 \end{bmatrix}, \quad \hat{r} = \begin{bmatrix} \hat{R}_1 \\ \dfrac{\partial \hat{R}_1}{\partial t} \end{bmatrix}, \quad \hat{d} = \begin{bmatrix} \hat{d}_1 \\ \hat{d}_2 \end{bmatrix}$$

$$\hat{y}_c = [\hat{y}_{c1}], \qquad C = [C_1, C_2] \tag{30}$$

where $a_{11}, a_{12}, a_{21}, b_{11}, b_{12}, b_{21}, b_{22}$ are time-invariant functions. Sufficient conditions for the controllability of this vortex system at a certain wave number β are

$$\det B(\beta) = b_{11}b_{22} - b_{12}b_{21} \neq 0$$

or

$$a_{21}b_{11}^2 - 2a_{11}b_{11}b_{21} - a_{12}b_{21}^2 \neq 0 \tag{31}$$

and the condition for observability of this vortex system at a certain wave number β, by measuring one parameter $\hat{y}_{c1}$ only, is

$$\det Q(\beta) = a_{12}C_1^2 - 2a_{11}C_1C_2 - a_{21}C_2^2 \neq 0 \tag{32}$$

It has to be emphasized at this point that the analyses of most of the above-mentioned two-dimensional problems[3-9] and of all the above-mentioned three-dimensional problems[10-16] can be reduced to the solution of the two-rank problems described by Eqs. (24) and (30).

Active Control of Vortex Stability

The new control-theory approach to vortex-system stability that was just presented introduces a new possibility of actively controlling vortex stability by two basic classic methods.

The first method of controlling a vortex-system stability is in an open-loop mode. This method is derived directly from Eq. (24) and uses an a priori known surface oscillation $R_1(x_R,t)$ to control the development of the disturbances along the vortex system. When the basic vortex state is steady, or quasisteady, a Laplace transform of Eq. (24) gives the following transfer function for each β:

$$\hat{y}(\beta,S) = \left[SI - A(\beta)\right]^{-1}\left\{B(\beta)\left[\tfrac{1}{S}\right]\hat{R}_1(\beta,S) + \hat{y}_0(\beta) + \hat{d}(\beta,S)\right\} \tag{33}$$

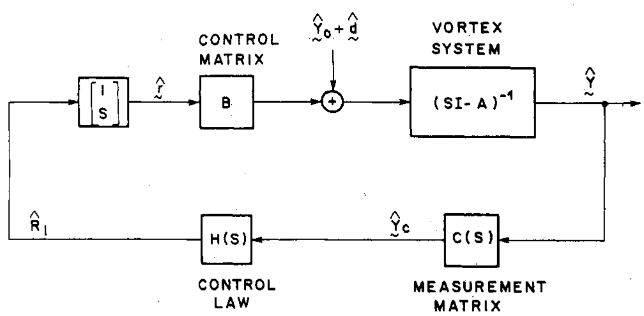

Fig. 2 Closed-loop feedback control.

where S is the Laplace transform variable. A description of the open-loop scheme in this case is shown in Fig. 1. Such a control method can be applied only to wave numbers β that are controllable according to the criteria previously defined.

The second control method is in a closed loop with a feedback mode (Fig. 2) that can be derived from Eqs. (24) and (27), together with an appropriate control law in the general form

$$\hat{R}_1(\beta,t) = H\left[\hat{y}_c(\beta,t)\right] \tag{34}$$

where H is a control operator that acts on the output vector $\hat{y}_c(\beta,t)$. In this method, the vortex disturbances $[\hat{y}(\beta,t)]$ are measured [Eq. (27)] at each wave number β. The resulting measured data $[\hat{y}_c(\beta,t)]$ are used by the control operator in a feedback loop to activate surface oscillations $\hat{R}_1(\beta,t)$ in order to control the vortex disturbances vector $\hat{y}(\beta,t)$.

When the basic vortex state is steady, or quasisteady, and when H is a time-invariant linear-control operator, a Laplace transform of Eqs. (24), (27), and (34) results, at each β, in

$$S\hat{y}(\beta,S) = A(\beta)\hat{y}(\beta,S) + B(\beta)\hat{r}(\beta,S) + \hat{y}_0(\beta) + \hat{d}(\beta,S)$$

$$\hat{y}_c(\beta,S) = C(\beta)\hat{y}(\beta,S)$$

$$\hat{R}_1(\beta,S) = H(\beta)\hat{y}_c(\beta,S)$$

$$\hat{r}(\beta,S) = \left[\tfrac{1}{S}\right]\hat{R}_1(\beta,S) \tag{35}$$

The controlled vortex disturbances are characterized by the eigenvalues of the controlled matrix

$$A_c(\beta) = \left[I - \boldsymbol{b}_2(\beta)H(\beta)C(\beta)\right]^{-1}\left[A(\beta) + \boldsymbol{b}_1(\beta)H(\beta)C(\beta)\right] \tag{36}$$

where $\boldsymbol{b}_1(\beta)$ and $\boldsymbol{b}_2(\beta)$ are the column vectors of matrix $B(\beta)$: $B(\beta) = [\boldsymbol{b}_1(\beta)|\boldsymbol{b}_2(\beta)]$.

The closed-loop feedback control can be applied only at wave numbers β that are controllable and observable according to the criteria previously defined.

Examples

The new control-theory approach and methods can now be applied to any two- or three-dimensional stability analysis of a straight-vortex system. In the present paper, the stability problems of a single vortex centered in a circular tube and that of a single vortex moving above a straight plane are considered. More problems, especially of vortex systems in or near a circular tube, were treated by the authors[17] and will be published in subsequent papers.

Line Vortex Concentric with a Circular Tube

In the case of a straight vortex filament with circulation Γ running down the center of a circular tube of radius R_0, it was shown by Rusak and Seginer[16] and Rusak[17] that

$$a_{11} = 0, \qquad a_{12} = -a_{21} = \frac{\Gamma}{2\pi R_0^2}\left[\beta^2 W(\delta) - \frac{\psi(\beta)\chi(\beta)}{1 - I_{c1}(\beta)}\right]$$

$$b_{11} = b_{12} = b_{21} = b_{22} = 0 \tag{37}$$

where δ is the dimensionless cutoff distance, $\delta = C^*\beta$, C^* is a vortex core-diameter parameter that can be evaluated by the Widnall[23] or Moore and Saffman[24] models of the cutoff distance, $W(\delta)$ is Crow's self-induced velocity function,[12] $\psi(\beta)$, $\chi(\beta)$ are Crow's interaction functions,[12] and $I_{c1}(\beta)$ is a function defined by Rusak and Seginer.[16]

Equations (37) show that the vortex in the center of a circular tube is dynamically stable at all wave numbers β.[16] Since $B \equiv 0$ in the linear approximation at every β (because of symmetry considerations), it can be concluded from Eq. (31) that the concentric vortex in a circular tube is not controllable at any β; therefore, it cannot be controlled by unsteady radial cross-section variations along the tube. This vortex is, however, observable by measuring one parameter only [Eq. (32)] at every β, except for $\beta_0(C^*)$ that satisfies

$$\beta_0^2 W(C^*\beta_0) = \frac{\psi(\beta_0)\chi(\beta_0)}{1 - I_{c1}(\beta_0)}$$

The behavior of β_0 vs C^* is shown in Ref.[16] Only when $\beta = \beta_0$ does full observability require the measurement of both parameters that define the perturbations to the vortex line.

Vortex near a Plane

A vortex with circulation Γ that is moving near a straight plane at a distance h has an inherent tendency toward the well-known symmetric Crow instability.[12] It can be shown[12,17] that in this case

$$a_{11} = 0, \qquad a_{12} = \frac{\Gamma}{2\pi h^2}\left[\beta^2 W(\delta) - \frac{1}{4} - \frac{\chi(2\beta)}{4}\right]$$

$$a_{21} = -\frac{\Gamma}{2\pi h^2}\left[\beta^2 W(\delta) + \frac{1}{4} - \frac{\psi(2\beta)}{4}\right]$$

$$b_{11} = \frac{\Gamma}{2\pi h^2} I_1(\beta), \qquad b_{12} = 0$$

$$b_{21} = \frac{i\beta U}{h} e^{-\beta}, \qquad b_{22} = e^{-\beta} \tag{38}$$

where

$$I_1(\beta) \equiv \frac{8}{\pi}\int_0^\infty \frac{\bar{y}^2}{(1+\bar{y}^2)^3}\,\chi\left(\beta\sqrt{1+\bar{y}^2}\right)\,\mathrm{d}\bar{y}$$

is an influence coefficient that cannot be expressed analytically, but can be evaluated numerically. $I_1(0) = 1/2$, for every β $I_1(\beta) > 0$, and as $\beta \to \infty$ $I_1(\beta)$ goes to zero.

The eigenvalues of the free (unforced, or $\hat{R}_1 \equiv 0$, $\hat{d} \equiv 0$) problem are

$$\lambda_{1,2} = -i\frac{\beta}{h}U \pm \sqrt{\bar{C}_{v0}^2(\beta)}$$

$$\bar{C}_{v0}^2(\beta) \equiv -\left(\frac{\Gamma}{2\pi h^2}\right)^2\left[\beta^2 W(\delta) + \frac{1}{4} - \frac{\psi(2\beta)}{4}\right] \times \left[\beta^2 W(\delta) - \frac{1}{4} - \frac{\chi(2\beta)}{4}\right] \tag{39}$$

The function $\bar{C}_{v0}^2(\beta)$ is presented in Fig. 3 for several values of the parameter C^*. The expression for $\bar{C}_{v0}^2$ is similar to the expression that was used by Crow[12] to investigate the dynamic stability of the development of perturbations along a pair of trailing vortices in their symmetric mode. It was found[12] that with $C^* < 0.2$, an instability can occur at wave numbers $0 \le \beta \le \beta_1(C^*)$, whereas for $\beta > \beta_1(C^*)$ the vortex pair is stable.

The vortex near a plane (or the Crow vortex pair in the symmetric mode) is observable. With $C_1 = 0$ and $C_2 = 1$ in the observability condition [Eq. (32)], a vortex near a parallel plane is fully observable at every $\beta > 0$ by measuring just one of its parameters, because $[\beta^2 W(\delta) + 1/4 - \psi(2\beta)/4] > 0$ for every positive wave number ($\beta > 0$) and for $C^* < 0.2$ (including the region where instability can occur). When $\beta = 0$ (a two-dimensional perturbation), full observability by one parameter only can be proven by Eq. (32) with $C_1 = 1$, $C_2 = 0$.

It can be concluded from Eq. (31) that since for every β,

$$\det B(\beta) = \frac{\Gamma}{2\pi h^2} I_1(\beta)e^{-\beta} \neq 0$$

then a single vortex filament that parallels a plane surface and is in its proximity is also controllable at every wave number β, including those at which instability may occur, and it can be

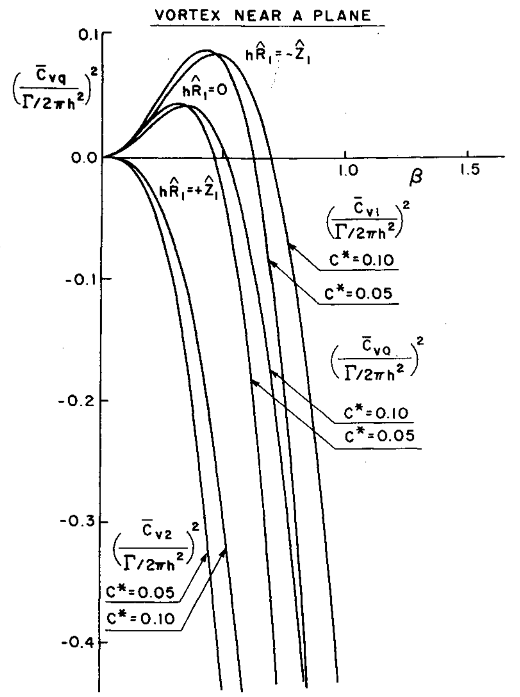

Fig. 3 Normalized generalized-damping coefficients for different control laws.

stabilized by an appropriate unsteady perturbation $R_1(x,t)$ to the solid planar surface.

Application of the closed-loop active control method (Fig. 2) with $C=(0,1)$ and $H=(-1)^q$ ($q=1$ or 2) results in the following control law for every wave number β:

$$\hat{R}_1(\beta,t)=(-1)^q\hat{z}_1(\beta,t) \qquad (q=1 \text{ or } 2) \tag{40}$$

Applying Eqs. (24), (30), (38), and (40) to this particular case results in the control matrix $A_c(\beta)$ with the following eigenvalues:

$$\lambda_{1,2}=-i\frac{\beta}{h}U\pm\sqrt{\bar{C}_{vq}^2(\beta)} \tag{41}$$

where

$$\bar{C}_{v1}^2(\beta)=-\left(\frac{\Gamma}{2\pi h^2}\right)^2\left[\beta^2W(\delta)+\frac{1}{4}-\frac{\psi(2\beta)}{4}\right]\times\frac{[\beta^2W(\delta)-¼-\chi(2\beta)/4-I_1(\beta)]}{1+e^{-\beta}}$$

$$\bar{C}_{v2}^2(\beta)=-\left(\frac{\Gamma}{2\pi h^2}\right)^2\left[\beta^2W(\delta)+\frac{1}{4}-\frac{\psi(2\beta)}{4}\right]\times\frac{[\beta^2W(\delta)-¼-\chi(2\beta)/4+I_1(\beta)]}{1-e^{-\beta}} \tag{42}$$

The functions $\bar{C}_{v1}^2(\beta)$ and $\bar{C}_{v2}^2(\beta)$ also are presented in Fig. 3 for several values of C^*. When $q=1$, the function $\bar{C}_{v1}^2(\beta)$ behaves similarly to $\bar{C}_{v0}^2$, but instability increases. It can occur in a wider range of wave numbers and has higher values of the exponential growth rate of the perturbations. On the other hand, when $q=2$, then $\bar{C}_{v2}^2(\beta)\le 0$ at every wave number β. This means that for every β, the development of the perturbations along the vortex can be fully stabilized by activating unsteady oscillations of the planar surface that obey the control law $\hat{R}_1=\hat{z}_1$.

To summarize, the well-known inherent symmetric Crow instability of a vortex moving near a planar surface[12,17] can be amplified or suppressed at will by using active lengthwise oscillations of the surface that obey the control law of Eq. (40). It should be emphasized that this control law requires the measurement of the development of only the vertical displacement perturbation of the vortex and the simultaneous activation of the surface oscillations. When the surface oscillations are in phase ($q=2$) with the vertical displacement of the vortex, full stabilization can be achieved, whereas when the surface oscillations are out of phase ($q=1$), instability is increased.

Conclusions

The linear three-dimensional equations that describe the linear stability of vortex systems near a parallel surface have the general form of the classic linear-control problem, with unsteady surface oscillations as the controller. The "controllability" and "observability" of vortex-system stability are defined by the classic criteria of linear-control theory. Criteria for the analysis of the control of vortex systems result in the proposed method of actively controlling the vortex stability. The methods of open-loop control and closed-loop with feedback control are presented for vortex systems. The examples shown demonstrate the powerful potential of the new combined theories of vortex stability and control. By using active solid-surface oscillations, a control law is proposed for the closed-loop method to suppress or to amplify at will the well-known Crow instability of a vortex moving near a straight plane.

References

[1]Polhamus, E. C., "Vortex Lift Research: Early Contributions and Some Current Challenges," NASA CP 2416, 1986.

[2]Rossow, V. J., "Wake Hazard Alleviation Associated with Roll Oscillations of Wake Generating Aircraft," *Journal of Aircraft*, Vol. 23, No. 6, 1986, pp. 484-491.

[3]Thomson, J. J., "On the Motion of Vortex Rings," Adams Prize Essay, London, 1883.

[4]Von Kármán, T., "Über den Mechanismus des Widerstands, den ein bewegter, Körper in einer Flüssigkeit erfährt," Göttinger Nachrichten, *Mathematische Physik,* Vol. K1, 1912, pp. 547-556.

[5]Föppl, L., "Wirbelbewegung hinter einen Kreiszylinder," *Sitzungsberichte der Bayerischen Akademie der Wissenschaften zu München,* Munich, Germany, 1913, pp. 1-17.

[6]Havelock, T. H., "The Stability of Motion of Rectilinear Vortices in Ring Formation," *Philosophical Magazine*, Series 7, Vol. 11, No. 70, Suppl., Feb. 1931, pp. 617-633.

[7]Lamb, Sir H., *Hydrodynamics*, 6th ed., Dover, New York, 1932, pp. 202-229.

[8]Seath, D. D., "Equilibrium Vortex Positions," *Journal of Spacecraft and Rockets*, Vol. 8, No. 1, 1971, pp. 72-75.

[9]Weihs, D., and Boasson, M., "Multiple Equilibrium Vortex Positions in Symmetric Shedding from Slender Bodies," *AIAA Journal*, Vol. 17, No. 2, 1979, pp. 213-214.

[10]Thomson, Sir W. (Lord Kelvin), "On the Vibrations of a Columnar Vortex," *Philosophical Magazine,* Vol. 5, No. 10, 1880, pp. 155-168.

[11]Rosenhead, L., "The Spread of Vorticity in the Wake Behind a Cylinder," *Proceedings of the Royal Society, Series A.*, Vol. 127, 1930, pp. 590-612.

[12]Crow, S. C., "Stability Theory for a Pair of Trailing Vortices," *AIAA Journal,* Vol. 8, No. 12, 1970, pp. 2172-2179.

[13]Jimenez, J., "Stability of a Pair of Co-Rotating Vortices," *Physics of Fluids*, Vol. 18, No. 11, 1975, pp. 1580-1581.

[14]Robinson, A. C., and Saffman, P. G., "Three-Dimensional Stability of Vortex Arrays," *Journal of Fluid Mechanics*, Vol. 125, No. 12, 1982, pp. 411-427.

[15]Widnall, S., "Review of Three-Dimensional Vortex Dynamics: Implications for the Computation of Separated and Turbulent Flow," *Studies of Vortex Dominated Flows*, edited by M. Y. Hussaini and M. D. Salas, Springer-Verlag, New York, 1985, pp. 16-32.

[16]Rusak, Z., and Seginer, A., "The Stability of a Vortex Filament in the Center of a Circular Tube," *29th Israel Annual Conference on Aviation and Astronautics*, Haifa, Israel, Feb. 1987, pp. 16-26.

[17]Rusak, Z., "Three-Dimensional Analysis of Equilibrium and Stability of Discrete Thin Vortices," Ph.D. Dissertation, Technion—Israel Inst. of Technology, Haifa, Israel, July 1989 (in Hebrew).

[18]Rusak, Z., and Seginer, A., "A General Theory of the Three-Dimensional Long-Wave Linear Stability of Systems of Straight Vortices," Rensselaer Polytechnic Inst. Math. Rept. 183, Troy, NY, Feb. 1990.

[19]Rusak, Z., and Seginer, A., "Dynamic and Static (Divergence) Stability of Vortex Systems," *30th Israel Annual Conference on Aviation and Astronautics*, Haifa, Israel, Feb. 1989, pp. 51–61.

[20]Rusak, Z., and Seginer, A., "The Controllability and Observability of Vortex Systems Stability," *30th Israel Annual Conference on Aviation and Astronautics*, Haifa, Israel, Feb. 1989, pp. 173–180.

[21]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972, Chap. 2.

[22]Batchelor, G. K., *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge, England, UK, 1967, pp. 85, 93.

[23]Widnall, S. E., "The Structure and Dynamics of Vortex Filaments," *Annual Review of Fluid Mechanics*, Vol. 7, 1975, pp. 141–165.

[24]Moore, D. W., and Saffman, P. G., "The Motion of a Vortex Filament with Axial Flow," *Philosophical Transactions of the Royal Society of London*, Vol. A272, No. 1226, 1972, pp. 403–429.